Some notes about simplicial complexes and homology II

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Simplicial Complexes

Definition

Let V be an ordered set, called the vertex set.
A simplex over V is any finite subset of V.

Definition

Let α and β be simplices over V, we say α is a face of β if α is a subset of β.

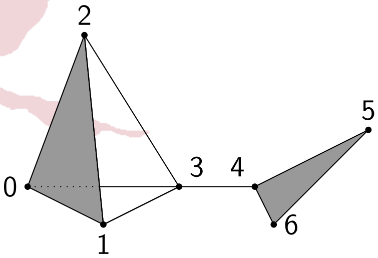
Definition

An ordered (abstract) simplicial complex over V is a set of simplices $\mathcal{K}$ over V satisfying the property:

$$\forall \alpha \in \mathcal{K}, \text{ if } \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K}$$

Let $\mathcal{K}$ be a simplicial complex. Then the set $S_n(\mathcal{K})$ of n-simplices of $\mathcal{K}$ is the set made of the simplices of cardinality $n+1$.

Simplicial Complexes



$$V = (0, 1, 2, 3, 4, 5, 6)$$
$$\mathcal{K} = \{\emptyset, (0), (1), (2), (3), (4), (5), (6),$$

$$(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3), (3, 4), (4, 5), (4, 6), (5, 6),$$
$$(0, 1, 2), (4, 5, 6)\}$$

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Chain Complexes

Definition

A chain complex C_* is a pair of sequences $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ where:

- *For every $q \in \mathbb{Z}$, the component C_q is an R-module (the case $R = \mathbb{Z}$ is the most common case in Algebraic Topology, we consider this case from now on), the chain group of degree q*
- *For every $q \in \mathbb{Z}$, the component d_q is a module morphism $d_q : C_q \to C_{q-1}$, the differential map*
- *For every $q \in \mathbb{Z}$, the composition $d_q d_{q+1}$ is null: $d_q d_{q+1} = 0$*

Definition

If $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ is a chain complex:

- *The image $B_q = \text{im } d_{q+1} \subseteq C_q$ is the (sub)module of q-boundaries*
- *The kernel $Z_q = \ker d_q \subseteq C_q$ is the (sub)module of q-cycles*

Homology

Given a chain complex $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$:

- $d_{q-1} \circ d_q = 0 \Rightarrow B_q \subseteq Z_q$
- Every boundary is a cycle
- The converse is not generally true

Homology

Given a chain complex $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$:

- $d_{q-1} \circ d_q = 0 \Rightarrow B_q \subseteq Z_q$
- Every boundary is a cycle
- The converse is not generally true

Definition

Let $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ be a chain complex. For each degree $n \in \mathbb{Z}$, the n-homology module of C_* is defined as the quotient module

$$H_n(C_*) = \frac{Z_n}{B_n}$$

From Simplicial Complexes to Chain Complexes

Definition

Let $\mathcal{K}$ be an (ordered abstract) simplicial complex. Let $n \geq 1$ and $0 \leq i \leq n$ be two integers n and i. Then the face operator ∂_i^n is the linear map $\partial_i^n : S_n(\mathcal{K}) \to S_{n-1}(\mathcal{K})$ defined by:

$$\partial_i^n(\{v_0, \ldots, v_n\}) = \{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$$

the i-th vertex of the simplex is removed, so that an $(n-1)$-simplex is obtained.

Definition

Let $\mathcal{K}$ be a simplicial complex. Then the chain complex $C_*(\mathcal{K})$ canonically associated with $\mathcal{K}$ is defined as follows. The chain group $C_n(\mathcal{K})$ is the free $\mathbb{Z}$ module generated by the n-simplices of $\mathcal{K}$. In addition, let $\{v_0, \ldots, v_{n-1}\}$ be a n-simplex of $\mathcal{K}$, the differential of this simplex is defined as:

$$d_n := \sum_{i=0}^{n} (-1)^i \partial_i^n$$

Definition

Let $\mathcal{K}$ be a simplicial complex. Then, the n-homology group of $\mathcal{K}$ is defined as:

$$H_n(K) = H_n(C_n(\mathcal{K}))$$

Butterfly example

- C_0, the free $\mathbb{Z}$-module on the set $\{(0),(1),(2),(3),(4),(5),(6)\}$.
- C_1, the free $\mathbb{Z}$-module on the set $\{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3),(3,4),(4,5),(4,6),(5,6)\}$.
- C_2, the free $\mathbb{Z}$-module on the set $\{(0,1,2),(4,5,6)\}$.

and the differential is provided by:

- $d_0((i)) = 0$,
- $d_1((i \;\; j)) = j - i$,
- $d_2((i \;\; j \;\; k)) = (j \;\; k) - (i \;\; k) + (i \;\; j)$.

and it is extended by linearity to the combinations $c = \sum_{i=1}^{m} \lambda_i x \in C_n$ where $\lambda_i \in \mathbb{Z}$ and $x \in C_n$.

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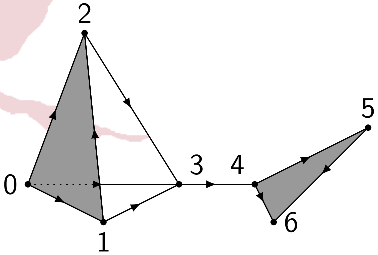
Differential Matrices

If the chain complex has a finite number of generators, we can represent the differential maps by means of finite matrices

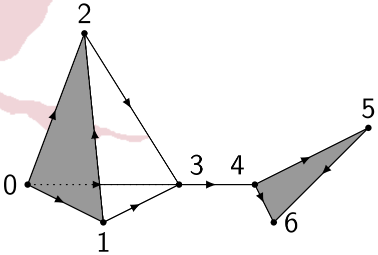
It is worth noting that incidence matrices are a particular case of these matrices

Differential Matrices



$$d_1 = \begin{array}{cc} & \begin{array}{cccccccccc} \{0,1\} & \{0,2\} & \{0,3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{3,4\} & \{4,5\} & \{4,6\} & \{5,6\} \end{array} \\ \begin{array}{c} \{0\} \\ \{1\} \\ \{2\} \\ \{3\} \\ \{4\} \\ \{5\} \\ \{6\} \end{array} & \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Differential Matrices



$$d_2 = \begin{array}{c} \\ \{0,1\} \\ \{0,2\} \\ \{0,3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{3,4\} \\ \{4,5\} \\ \{4,6\} \\ \{5,6\} \end{array} \begin{array}{c} \begin{array}{cc} \{0,1,2\} & \{4,5,6\} \end{array} \\ \left(\begin{array}{cc} 1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{array} \right) \end{array}$$

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Computing homology groups from Smith Normal Form

Let C_* be a *finite* chain complex and d_n, d_{n+1} be the differential maps of X of dimension n and $n+1$.

If we compute the Smith Normal Form of both matrices we obtain two matrices of the form:

$$SNF(d_n) = \begin{pmatrix} a_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_k & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} SNF(d_{n+1}) \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & b_i & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & b_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

Then $H_n(X) = \mathbb{Z}_{b_i} \oplus \mathbb{Z}_{b_{i+1}} \oplus \dots \mathbb{Z}_{b_m} \oplus \mathbb{Z}^{f-k-m}$ where f is the number of generators of C_* of dimension n

Butterfly Example

Let us compute H_0 of the butterfly simplicial complex
So, we need M_0 and M_1:

- in this case M_0 is the void matrix; so $k = 0$;
- we compute the Smith Normal Form of M_1:

$$SNF(d_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

  so, $m = 6$;
- in addition, there are 7 0-simplexes

Therefore, $H_0(X) = \mathbb{Z}^{7-6-0} = \mathbb{Z}$

This result must be interpreted as stating that the butterfly simplicial complex only has one connected component

Butterfly Example continued

Let us compute H_1 of the butterfly simplicial complex
So, we need M_1 and M_2:

- we have computed in the previous slide the Smith Normal Form of M_1: $k = 6$;
- we compute the Smith Normal Form of M_2:

$$SNF(d_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};$$

  so, $m = 2$;
- in addition, there are 10 1-simplexes

Therefore, $H_1(X) = \mathbb{Z}^{10-6-2} = \mathbb{Z} \oplus \mathbb{Z}$
This result must be interpreted as stating that the butterfly simplicial complex has two "holes" in the topological sense
You can think that there is three holes in the butterfly example, but one of them is the composition of the others

A more detailed explanation about this fact is given in Page 6 of `http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Genova-Lecture-Notes.pdf`

Butterfly Example continued

Let us compute H_2 of the butterfly simplicial complex
So, we need M_2 and M_3:

- we have computed in the previous slide the Smith Normal Form of M_2: $k = 2$;
- M_3 is a void matrix; so, $m = 0$,
- in addition, there are 2 2-simplexes

Therefore, $H_2(X) = \mathbb{Z}^{2-2-0} = 0$
This result must be interpreted as stating that the butterfly simplicial complex has not "voids" in the topological sense

The rest of matrices are void, then the homology groups $H_n(X)$ with $n \geq 3$ are null

Other Example

Consider the matrices:

$$d_n = \begin{pmatrix} 0 & 0 & 2 & 0 \end{pmatrix} \; d_{n+1} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 \end{pmatrix}$$

$$SNF(d_n) = \begin{pmatrix} 2 & 0 & 0 & 0 \end{pmatrix} \; SNF(d_{n+1}) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$H_n = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^{4-2-1} = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}$