A Certified Reduction Strategy for Homological Image Processing*

M. Poza, C. Domínguez, J. Heras, and J. Rubio

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PROLE'14

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Goal

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A formally-verified and efficient method to analyse digital images.

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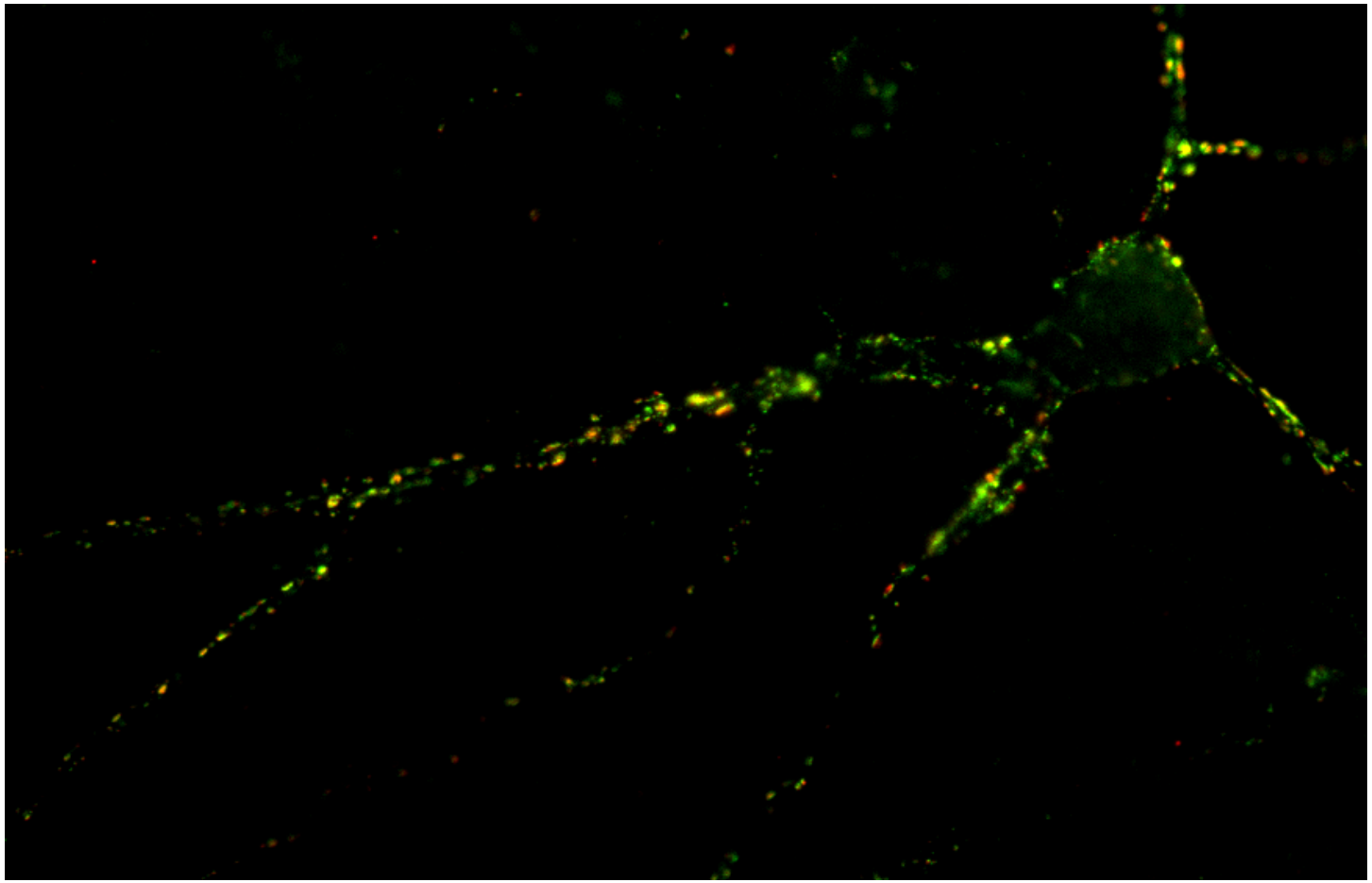
A motivating example: counting synapses

- *Synapses* are the points of connection between neurons.
- *Relevance*: Computational capabilities of the brain.
- Procedures to modify the synaptic density may be an important asset in the treatment of neurological diseases.
- An automated, efficient, and reliable method is necessary.

Counting Synapses



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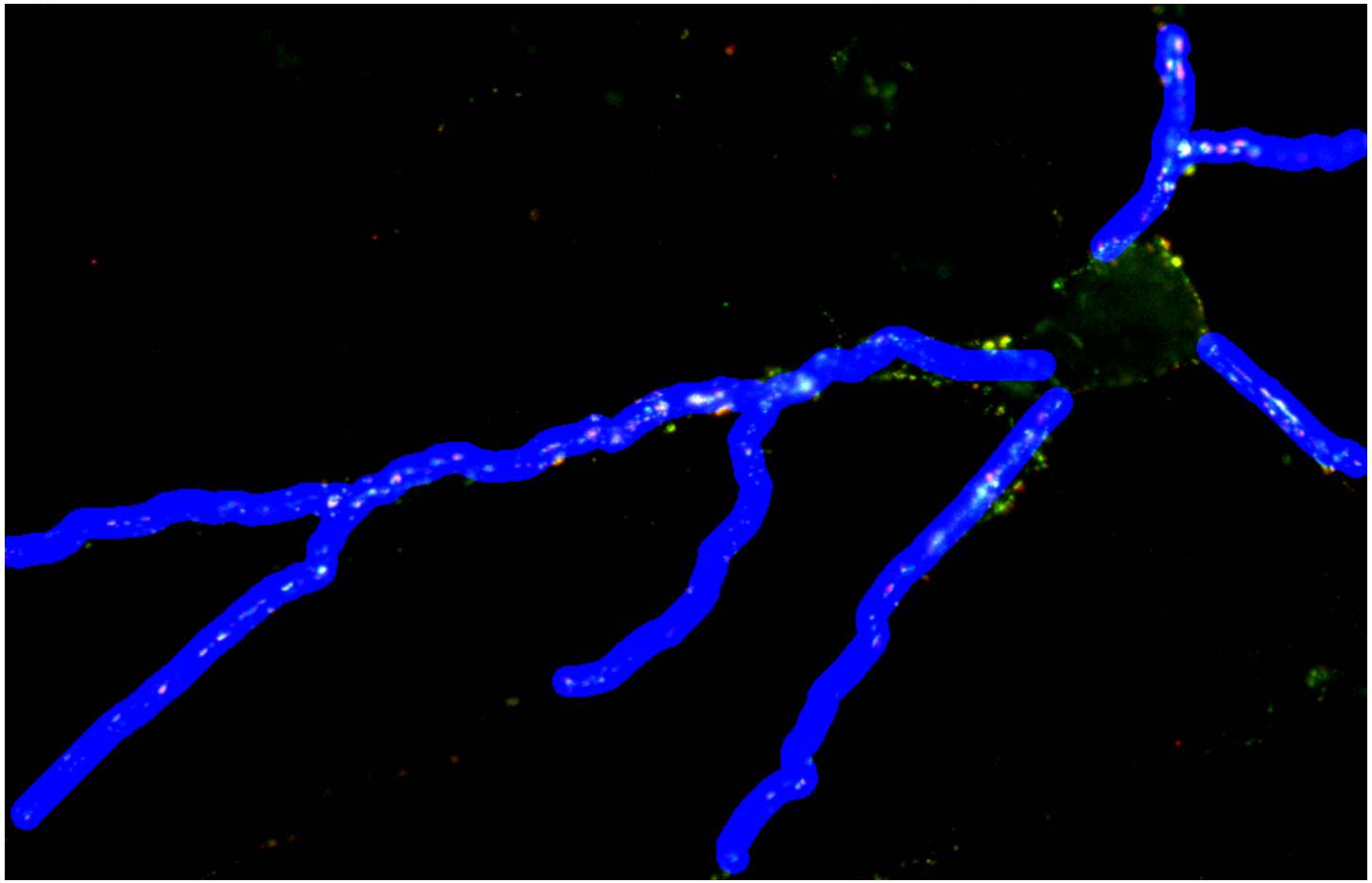
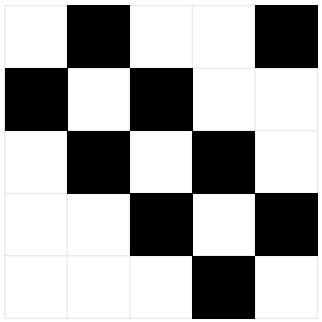


Counting Synapses

Algebraic Topology and Digital Images

Digital Image



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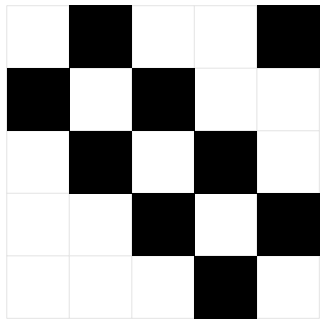

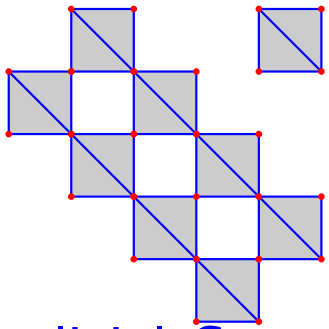

Simplicial Complex

Algebraic Topology and Digital Images

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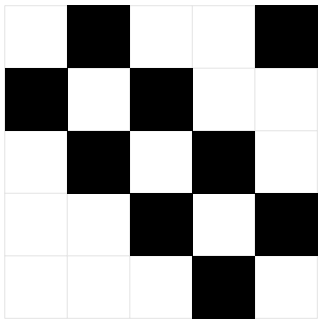

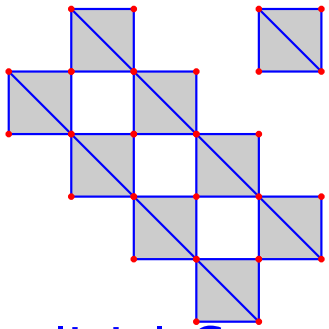

Simplicial Complex

$C_0 =$ vertices
$C_1 =$ edges
$C_2 =$ triangles

Chain Complex

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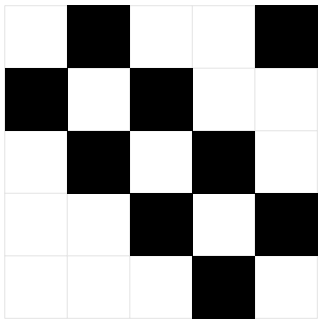

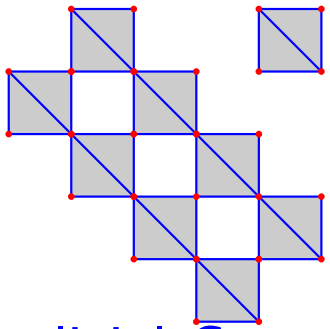

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Homology groups

$$H_0 = \mathbb{Z} \oplus \mathbb{Z}$$
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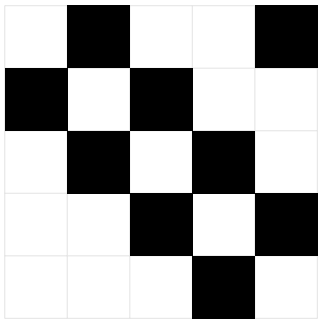

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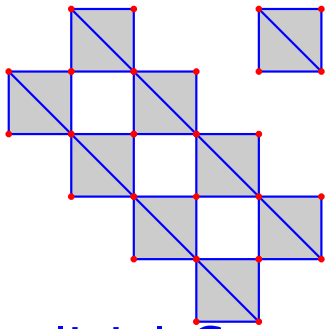

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Chain Complex

Interactive Proof Assistants

- What is an Interactive Proof Assistant?
  - Software tool for the development of formal proofs.
  - Man-Machine collaboration:
    - Human: design the proofs.
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  - Examples: Isabelle, Hol, ACL2, Coq, ...

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  - Examples: Isabelle, Hol, ACL2, Coq, ...
- Applications:
  - Mathematical proofs:
    - Four Colour Theorem.
    - Feit-Thompson Theorem.
    - Kepler conjecture.
  - Software and Hardware verification:
    - C compiler.
    - AMD5K86 microprocessor.
    - seL4: operating-system kernel.

Coq/SSReflect

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  - Interesting feature: program extraction from a constructive proof.
- SSReflect:
  - Extension of Coq.
  - Developed while formalising the Four Colour Theorem by G. Gonthier.
  - It has been used in the formalisation of the Feit-Thompson Theorem.

Problem

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A formally-verified (using Coq) and efficient method (from Algebraic Topology) to analyse digital images.

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Demo.

Outline

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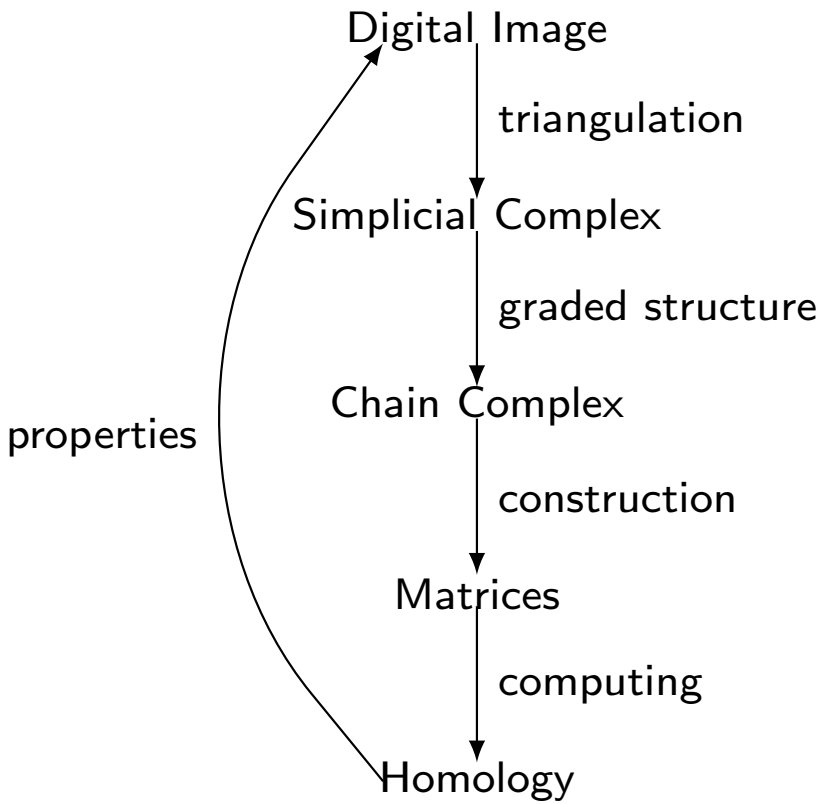
General Idea



J. Heras, M. Dénès, G. Mata, A. Mörtberg, M. Poza, and V. Siles. Towards a certified computation of homology groups. In proceedings 4th International Workshop on Computational Topology in Image Context. Lecture Notes in Computer Science, 7309, pages 49–57, 2012.

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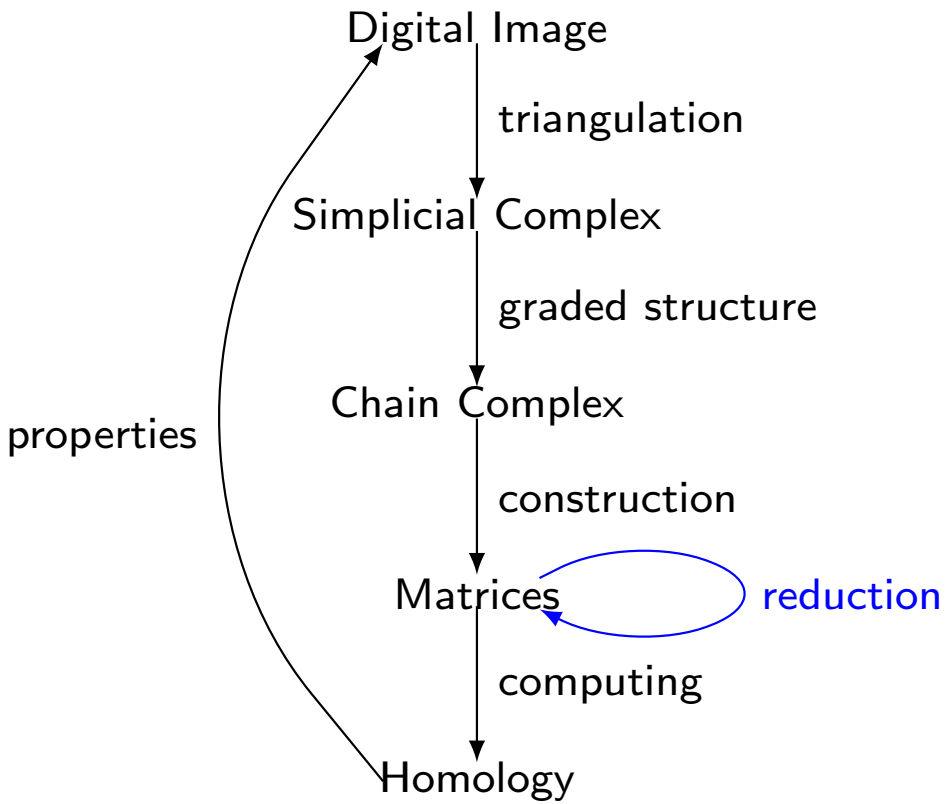

Concrete Goal

Formalise an algorithm that reduces the information but keeps the homological properties.

Discrete Morse Theory

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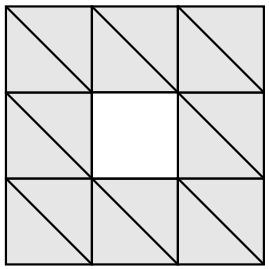

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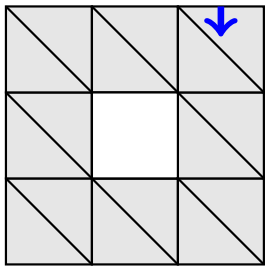

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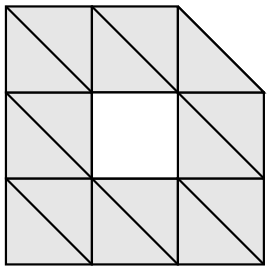

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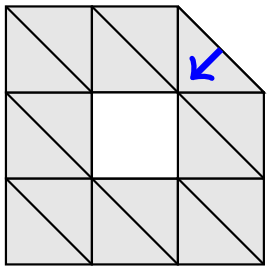

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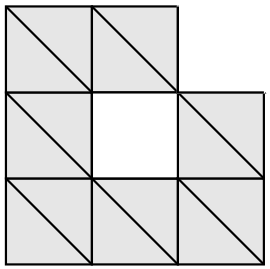

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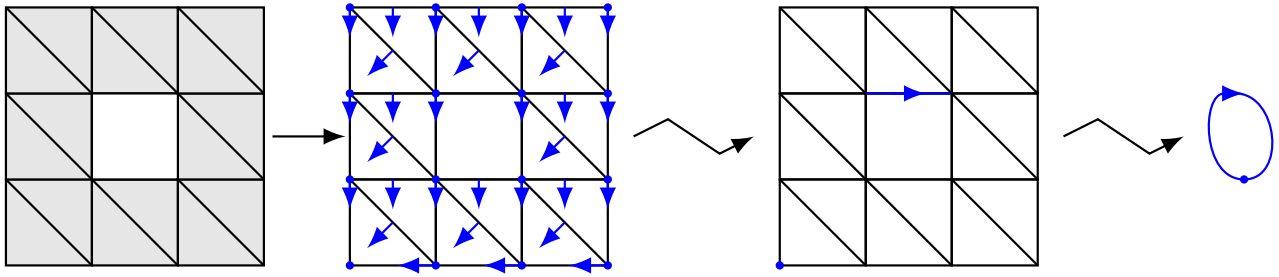

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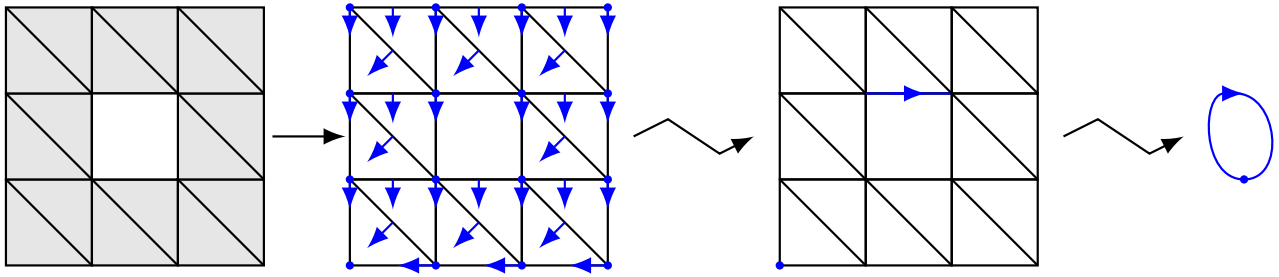

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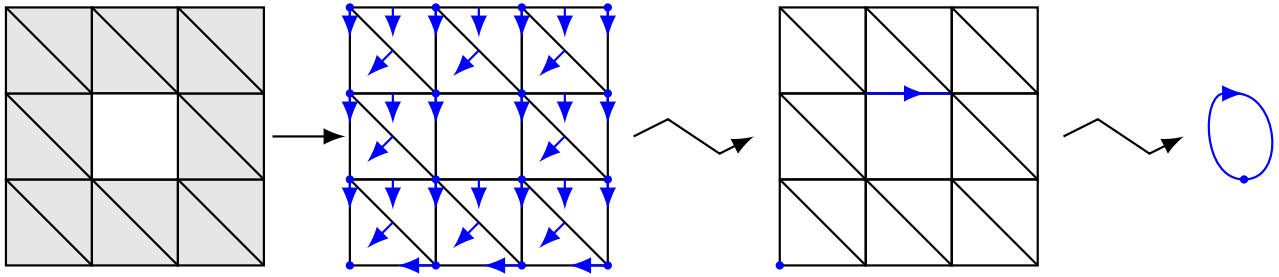

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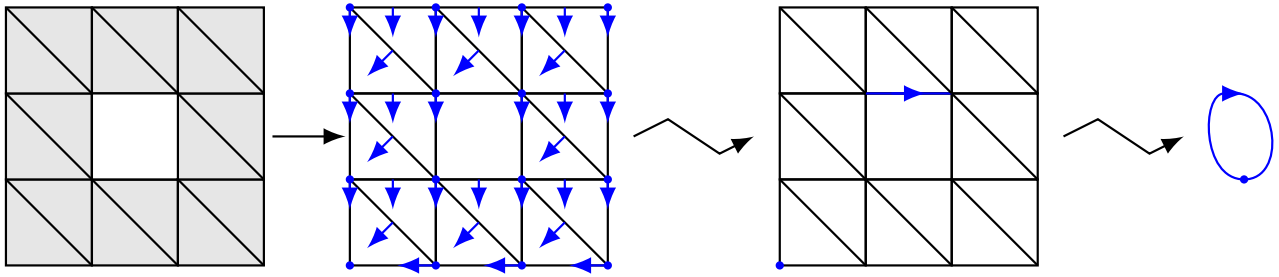

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- Compute an admissible discrete vector field *dvf*.
- Use *dvf* to reduce the matrices:
  - Vertex-Edge matrix: $16 \times 32 \Rightarrow 1 \times 1$.
  - Edge-Triangle matrix: $32 \times 16 \Rightarrow 1 \times 0$.

Formalised Results

Two main results have been formalised:

- The Coq's implementation of Romero-Sergeraert's is correct.
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Coq development about homology of digital images:

- 3,800 lines of code, 260 theorems, and 200 definitions.

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Fast Proof Prototyping

3rd Workshop on Mechanizing Metatheory (2008)

"The use of proof assistants in programming language research is still not commonplace: the available tools are confusingly diverse, difficult to learn, inadequately documented, and lacking in specific library facilities required for work in programming languages."

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Proof assistants have improved in the last few years, but they are not there yet.

From computation to verification through testing

- *Haskell* as programming language.
- *QuickCheck* to test the programs.
- *Coq/SSReflect* to verify the correctness of the programs.

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Haskell:

Algorithm (*gen_adm_dvf*)

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Coq/SSReflect:

SSReflect Theorem:

```
Theorem gen_adm_dvf_is_admissible (M : seq (seq Z2)) :
 admissible (gen_adm_dvf M).
```

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500 randomly generated matrices:

- Initial size of the matrices: 100×300.
- Time: 12 seconds.

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Problem

For bigger matrices, the reduction process takes a lot of time.

Solution

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Example (Inverse of a matrix):

- Problem related to handling big datastructures.
- Given a matrix M, compute (in Haskell) its inverse matrix M_1.
- Prove in Coq that $M \times M_1 = 1$ (`by compute`).
- Continue the computation in Coq.

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Conclusions and Further Work

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- Conclusions:
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- Further work:
  - Obtain better performance:
    - Improve algorithms.
    - Implement more efficient data structures.
    - Improve the running environments in proof assistants.
    - Extract certified code from Coq.
  - Apply methodology and techniques to:
    - Persistent homology and Zigzag persistence.
    - Application: automate the location of neuronal structure from a stack of images.

Bibliography about this work


M. Poza (2013): Certifying homological algorithms to study biomedical images. Ph.D. thesis, University of La Rioja.


M. Poza, C. Domínguez, J. Heras, and J. Rubio (2014): A certified reduction strategy for homological image processing. ACM Transactions on Computational Logic 15(3).


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