Incidence Simplicial Matrices Formalized in Coq/SSReflect*

Jónathan Heras, María Poza, Maxime Dénès, and Laurence Rideau

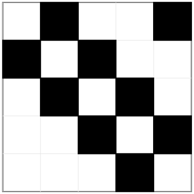
Universidad de La Rioja, Spain - INRIA Sophia Antipolis (Méditerranée)

CICM 2011, Calculemus track,
July 22, 2011

*Partially supported by Ministerio de Educación y Ciencia, project MTM2009-13842-C02-01, and by European Commission FP7, STREP project ForMath, n. 243847

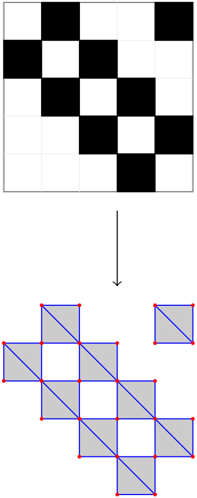
Algebraic Topology and Digital Images

Digital Image



Algebraic Topology and Digital Images

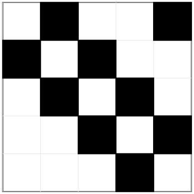
Digital Image

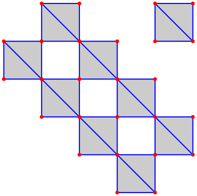


Simplicial complex

Algebraic Topology and Digital Images

Digital Image





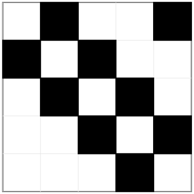
Simplicial complex

$$C_0 = \mathbb{Z}[vertices]$$
$$C_1 = \mathbb{Z}[edges]$$
$$C_2 = \mathbb{Z}[triangles]$$

Chain complex

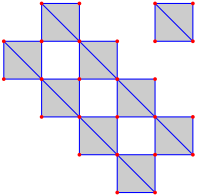
Algebraic Topology and Digital Images

Digital Image



Homology groups

$$H_0 = \mathbb{Z} \oplus \mathbb{Z}$$
$$H_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$



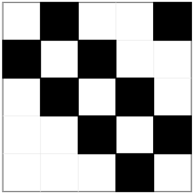
$$C_0 = \mathbb{Z}[vertices]$$
$$C_1 = \mathbb{Z}[edges]$$
$$C_2 = \mathbb{Z}[triangles]$$

Simplicial complex

Chain complex

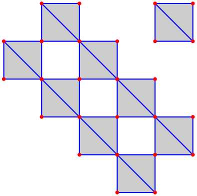
Algebraic Topology and Digital Images

Digital Image



Homology groups

$$H_0 = \mathbb{Z} \oplus \mathbb{Z}$$
$$H_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$



Simplicial complex

$$C_0 = \mathbb{Z}[vertices]$$
$$C_1 = \mathbb{Z}[edges]$$
$$C_2 = \mathbb{Z}[triangles]$$

Chain complex

Goal

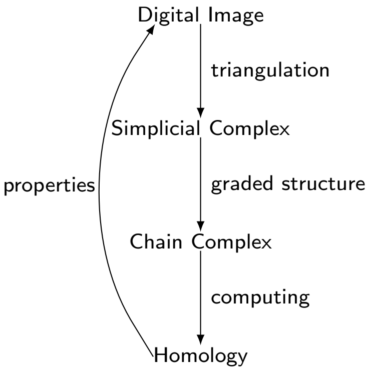
$$
\begin{tikzcd}
\text{Digital Image} \arrow[d, "\text{triangulation}"] \\
\text{Simplicial Complex} \arrow[d, "\text{graded structure}"] \\
\text{Chain Complex} \arrow[d, "\text{computing}"] \\
\text{Homology} \arrow[uuu, bend left, "\text{properties}"]
\end{tikzcd}
$$

Goal

$$
\begin{tikzcd}
\text{Digital Image} \arrow[d, "\text{triangulation}"] \\
\text{Simplicial Complex} \arrow[d, "\text{graded structure}"] \\
\text{Chain Complex} \arrow[d, "\text{computing}"] \\
\text{Homology} \arrow[uuu, bend left, "\text{properties}"]
\end{tikzcd}
$$

- Implemented in the Kenzo system

Goal

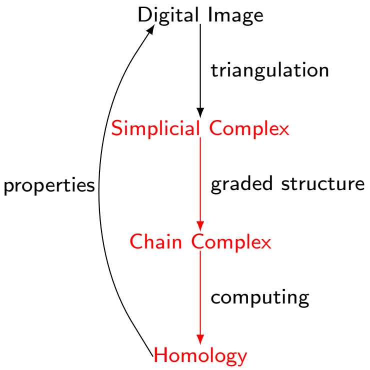


- Implemented in the Kenzo system

General Goal

Formalizing the computation of homology groups of digital images

Goal



- Implemented in the Kenzo system

General Goal

Formalizing the computation of homology groups of digital images

Table of Contents

Table of Contents

Digital Images

Digital Image → Simplicial Complex → Chain Complex → Homology

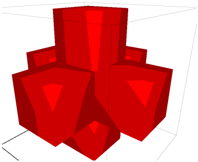
- 2D digital images:
    - elements are pixels

Digital Images

Digital Image $\longrightarrow$ Simplicial Complex $\longrightarrow$ Chain Complex $\longrightarrow$ Homology

- 2D digital images:
  - elements are pixels

- 3D digital images:
  - elements are voxels



Simplicial Complexes

Digital Image ⟶ **Simplicial Complex** ⟶ Chain Complex ⟶ Homology

Definition

Let V be an ordered set, called the vertex set.
A simplex over V is any finite subset of V.

Simplicial Complexes

Digital Image → Simplicial Complex → Chain Complex → Homology

Definition

Let V be an ordered set, called the vertex set.
A simplex over V is any finite subset of V.

Definition

Let α and β be simplices over V, we say α is a face of β if α is a subset of β.

Simplicial Complexes

Digital Image → **Simplicial Complex** → Chain Complex → Homology

Definition

Let V be an ordered set, called the vertex set.
A simplex over V is any finite subset of V.

Definition

Let α and β be simplices over V, we say α is a face of β if α is a subset of β.

Definition

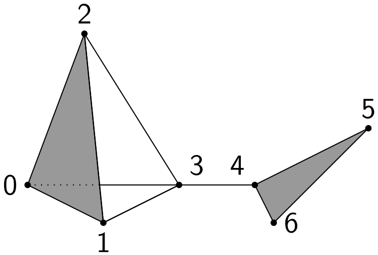
An ordered (abstract) simplicial complex over V is a set of simplices $\mathcal{K}$ over V satisfying the property:

$$\forall \alpha \in \mathcal{K}, \text{ if } \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K}$$

Let $\mathcal{K}$ be a simplicial complex. Then the set $S_n(\mathcal{K})$ of n-simplices of $\mathcal{K}$ is the set made of the simplices of cardinality $n+1$.

Simplicial Complexes

Digital Image → **Simplicial Complex** → Chain Complex → Homology



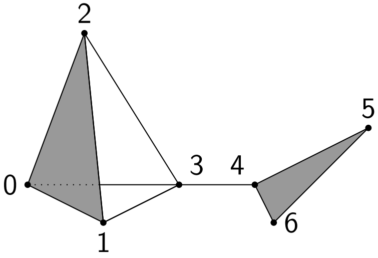
$V = (0, 1, 2, 3, 4, 5, 6)$
$\mathcal{K} = \{\emptyset, (0), (1), (2), (3), (4), (5), (6),$
$(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3), (3, 4), (4, 5), (4, 6), (5, 6),$
$(0, 1, 2), (4, 5, 6)\}$

Simplicial Complexes

Digital Image → Simplicial Complex → Chain Complex → Homology

Definition

The facets of a simplicial complex $\mathcal{K}$ are the maximal simplices of the simplicial complex.



The facets are: $\{(0,3),(1,3),(2,3),(3,4),(0,1,2),(4,5,6)\}$

Chain Complexes

Digital Image → Simplicial Complex → Chain Complex → Homology

Definition

A chain complex C_* is a pair of sequences $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ where:

- *For every $q \in \mathbb{Z}$, the component C_q is an R-module, the chain group of degree q*
- *For every $q \in \mathbb{Z}$, the component d_q is a module morphism $d_q : C_q \to C_{q-1}$, the differential map*
- *For every $q \in \mathbb{Z}$, the composition $d_q d_{q+1}$ is null: $d_q d_{q+1} = 0$*

Homology

Digital Image ⟶ Simplicial Complex ⟶ Chain Complex ⟶ **Homology**

Definition

If $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$ is a chain complex:

- *The image $B_q = \text{im } d_{q+1} \subseteq C_q$ is the (sub)module of q-boundaries*
- *The kernel $Z_q = \ker\, d_q \subseteq C_q$ is the (sub)module of q-cycles*

Given a chain complex $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$:

- $d_{q-1} \circ d_q = 0 \Rightarrow B_q \subseteq Z_q$
- Every boundary is a cycle
- The converse is not generally true

Homology

Digital Image ⟶ Simplicial Complex ⟶ Chain Complex ⟶ **Homology**

Definition

If $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$ is a chain complex:

- *The image $B_q = im\ d_{q+1} \subseteq C_q$ is the (sub)module of q-boundaries*
- *The kernel $Z_q = ker\ d_q \subseteq C_q$ is the (sub)module of q-cycles*

Given a chain complex $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$:

- $d_{q-1} \circ d_q = 0 \Rightarrow B_q \subseteq Z_q$
- Every boundary is a cycle
- The converse is not generally true

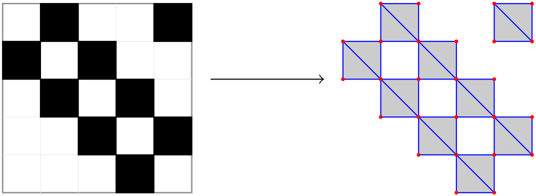
Definition

Let $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$ be a chain complex. For each degree $n \in \mathbb{Z}$, the n-homology module of C_* is defined as the quotient module

$$H_n(C_*) = \frac{Z_n}{B_n}$$

From a digital image to a simplicial complex

Digital Image ⟶ Simplicial Complex ⟶ Chain Complex ⟶ Homology



From Simplicial Complexes to Chain Complexes

Digital Image $\longrightarrow$ Simplicial Complex $\longrightarrow$ Chain Complex $\longrightarrow$ Homology

Definition

Let $\mathcal{K}$ be an (ordered abstract) simplicial complex. Let $n \geq 1$ and $0 \leq i \leq n$ be two integers n and i. Then the face operator ∂_i^n is the linear map $\partial_i^n : S_n(\mathcal{K}) \rightarrow S_{n-1}(\mathcal{K})$ defined by:

$$\partial_i^n((v_0, \ldots, v_n)) = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n).$$

The i-th vertex of the simplex is removed, so that an $(n-1)$-simplex is obtained.

Definition

Let $\mathcal{K}$ be a simplicial complex. Then the chain complex $C_*(\mathcal{K})$ canonically associated with $\mathcal{K}$ is defined as follows. The chain group $C_n(\mathcal{K})$ is the free $\mathbb{Z}$ module generated by the n-simplices of $\mathcal{K}$. In addition, let $(v_0, \ldots, v_{n-1})$ be a n-simplex of $\mathcal{K}$, the differential of this simplex is defined as:

$$d_n := \sum_{i=0}^{n} (-1)^i \partial_i^n$$

Computing

Digital Image → Simplicial Complex → Chain Complex → Homology

- Computing Homology groups:
  - From a Chain Complex $(C_n, d_n)_{n \in \mathbb{Z}}$:
    - d_n can be expressed as matrices
    - Homology groups are obtained from a diagonalization process

Computing

Digital Image ⟶ Simplicial Complex ⟶ Chain Complex ⟶ Homology

- Computing Homology groups:
  - From a Chain Complex $(C_n, d_n)_{n\in\mathbb{Z}}$:
    - d_n can be expressed as matrices
    - Homology groups are obtained from a diagonalization process
  - Directly from the Simplicial Complex:
    - Incidence simplicial matrices
    - Homology groups are obtained from a diagonalization process

Computing

Digital Image ⟶ Simplicial Complex ⟶ Chain Complex ⟶ Homology

- Computing Homology groups:
  - From a Chain Complex $(C_n, d_n)_{n \in \mathbb{Z}}$:
    - d_n can be expressed as matrices
    - Homology groups are obtained from a diagonalization process
  - Directly from the Simplicial Complex:
    - Incidence simplicial matrices
    - Homology groups are obtained from a diagonalization process

Table of Contents

From Simplicial Complexes to Homology

$$
\begin{tikzcd}
 & \text{Simplicial Complex} \arrow[dl] \arrow[d, "\text{graded structure}"] \\
\text{Incidence Matrices} \arrow[dr] & \text{Chain Complex} \arrow[l, dotted, "\text{differential}"'] \arrow[d, "\text{computing}"] \\
 & \text{Homology}
\end{tikzcd}
$$

Incidence Matrices

Definition

Let X and Y be two ordered finite sets of simplices, we call incidence matrix to a matrix $m \times n$ where

$$m = \sharp|X| \ \wedge \ n = \sharp|Y|$$

$$M = \begin{array}{c c} & \begin{array}{ccc} Y[1] & \cdots & Y[n] \end{array} \\ \begin{array}{c} X[1] \\ \vdots \\ X[m] \end{array} & \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \end{array}$$

$$a_{i,j} = \begin{cases} 1 & \textit{if } X[i] \textit{ is a face of } Y[j] \\ 0 & \textit{if } X[i] \textit{ is not a face of } Y[j] \end{cases}$$

Incidence Matrices

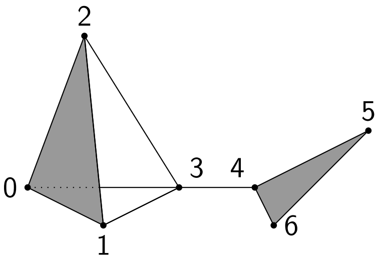
Definition

Let C be a finite set of simplices, A be the set of n-simplices of C with an order between its elements and B the set of $(n-1)$-simplices of C with an order between its elements.
We call incidence matrix of dimension n ($n \geq 1$), to a matrix $p \times q$ where

$$p = \sharp|B| \ \wedge \ q = \sharp|A|$$

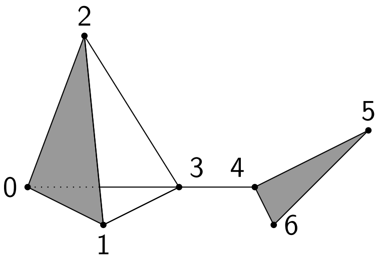
$$M_{i,j} = \begin{cases} 1 & \textit{if } B[i] \textit{ is a face of } A[j] \\ 0 & \textit{if } B[i] \textit{ is not a face of } A[j] \end{cases}$$

Incidence Matrices of Simplicial Complexes



$$
\begin{array}{c|c}
 & \begin{array}{cccccccccc} (0,1) & (0,2) & (0,3) & (1,2) & (1,3) & (2,3) & (3,4) & (4,5) & (4,6) & (5,6) \end{array} \\
\begin{array}{c} (0) \\ (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \end{array} &
\left(\begin{array}{cccccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)
\end{array}
$$

Incidence Matrices of Simplicial Complexes



$$
\begin{array}{c c}
 & \begin{array}{cc} (0,1,2) & (4,5,6) \end{array} \\
\begin{array}{c} (0,1) \\ (0,2) \\ (0,3) \\ (1,2) \\ (1,3) \\ (2,3) \\ (3,4) \\ (4,5) \\ (4,6) \\ (5,6) \end{array} &
\left(\begin{array}{cc}
1 & 0 \\
1 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{array}\right)
\end{array}
$$

Product of two consecutive incidence matrices

Theorem (Product of two consecutive incidence matrices)

Let $\mathcal{K}$ be a finite simplicial complex over V with an order between the simplices of the same dimension and let $n \geq 1$ be a natural number n, then the product of the n-th incidence matrix of $\mathcal{K}$ and the $(n+1)$-incidence matrix of $\mathcal{K}$ over the ring $\mathbb{Z}/2\mathbb{Z}$ is equal to the null matrix.

Sketch of the proof

- Let S_{n+1} be the set of $(n+1)$-simplices of $\mathcal{K}$ with an order between its elements
- Let S_n be the set of n-simplices of $\mathcal{K}$ with an order between its elements
- Let S_{n-1} be the set of $(n-1)$-simplices of $\mathcal{K}$ with an order between its elements

Sketch of the proof

- Let S_{n+1} be the set of $(n+1)$-simplices of $\mathcal{K}$ with an order between its elements
- Let S_n be the set of n-simplices of $\mathcal{K}$ with an order between its elements
- Let S_{n-1} be the set of $(n-1)$-simplices of $\mathcal{K}$ with an order between its elements

$$M_n(\mathcal{K}) = \begin{array}{c c} & \begin{array}{ccc} S_n[1] & \cdots & S_n[r1] \end{array} \\ \begin{array}{c} S_{n-1}[1] \\ \vdots \\ S_{n-1}[r2] \end{array} & \begin{pmatrix} a_{1,1} & \cdots & a_{1,r1} \\ \vdots & \ddots & \vdots \\ a_{r2,1} & \cdots & a_{r2,r1} \end{pmatrix} \end{array}, M_{n+1}(\mathcal{K}) = \begin{array}{c c} & \begin{array}{ccc} S_{n+1}[1] & \cdots & S_{n+1}[r3] \end{array} \\ \begin{array}{c} S_n[1] \\ \vdots \\ S_n[r1] \end{array} & \begin{pmatrix} b_{1,1} & \cdots & b_{1,r1} \\ \vdots & \ddots & \vdots \\ b_{r1,1} & \cdots & b_{r1,r3} \end{pmatrix} \end{array}$$

where $r1 = \sharp|S_n|$, $r2 = \sharp|S_{n-1}|$ and $r3 = \sharp|S_{n+1}|$

Sketch of the proof

$$M_n(\mathcal{K}) \times M_{n+1}(\mathcal{K}) = \begin{pmatrix} c_{1,1} & \cdots & c_{1,r3} \\ \vdots & \ddots & \vdots \\ c_{r2,1} & \cdots & c_{r2,r3} \end{pmatrix}$$

where

$$c_{i,j} = \sum_{1 \leq k \leq r1} a_{i,k} \times b_{k,j}$$

Sketch of the proof

$$M_n(\mathcal{K}) \times M_{n+1}(\mathcal{K}) = \begin{pmatrix} c_{1,1} & \cdots & c_{1,r3} \\ \vdots & \ddots & \vdots \\ c_{r2,1} & \cdots & c_{r2,r3} \end{pmatrix}$$

where

$$c_{i,j} = \sum_{1 \leq k \leq r1} a_{i,k} \times b_{k,j}$$

we need to prove that

$$\forall i, j, c_{i,j} = 0$$

in order to prove that $M_n \times M_{n+1} = 0$

Sketch of the proof

$$M_n(\mathcal{K}) \times M_{n+1}(\mathcal{K}) = \begin{pmatrix} c_{1,1} & \cdots & c_{1,r3} \\ \vdots & \ddots & \vdots \\ c_{r2,1} & \cdots & c_{r2,r3} \end{pmatrix}$$

where

$$c_{i,j} = \sum_{1 \leq k \leq r1} a_{i,k} \times b_{k,j}$$

we need to prove that

$$\forall i, j, c_{i,j} = 0$$

in order to prove that $M_n \times M_{n+1} = 0$
Since k enumerates the indices of elements of S_n:

$$c_{i,j} = \sum_{X \in S_n} F(S_{n-1}[i], X) \times F(X, S_{n+1}[j]) \text{ with } F(Y, Z) = \begin{cases} 1 & \text{if } Y \in dZ \\ 0 & \text{otherwise} \end{cases}$$

where

$$dZ = \{Z \setminus \{x\} \mid x \in Z\}$$

Sketch of the proof

$$c_{i,j} = \sum_{X \in S_n} F(S_{n-1}[i], X) \times F(X, S_{n+1}[j])$$

Sketch of the proof

$$
\begin{aligned}
c_{i,j} &= \sum_{X \in S_n} F(S_{n-1}[i], X) \times F(X, S_{n+1}[j]) \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 1 \\
&\quad + \sum_{X \in S_n | X \notin \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 0 \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X)
\end{aligned}
$$

Sketch of the proof

$$
\begin{aligned}
c_{i,j} &= \sum_{X \in S_n} F(S_{n-1}[i], X) \times F(X, S_{n+1}[j]) \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 1 \\
&\quad + \sum_{X \in S_n | X \notin \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 0 \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X) \\
&= \sum_{x \in S_{n+1}[j]} F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\})
\end{aligned}
$$

Sketch of the proof

$$
\begin{aligned}
c_{i,j} &= \sum_{X \in S_n} F(S_{n-1}[i], X) \times F(X, S_{n+1}[j]) \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 1 \\
&\quad + \sum_{X \in S_n | X \notin \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 0 \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X) \\
&= \sum_{x \in S_{n+1}[j]} F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) \\
&= \sum_{x \in S_{n+1}[j] | x \in S_{n-1}[i]} F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) + \\
&\quad \sum_{x \in S_{n+1}[j] | x \notin S_{n-1}[i]} F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\})
\end{aligned}
$$

Sketch of the proof

$$
\begin{aligned}
c_{i,j} &= \sum_{X \in S_n} F(S_{n-1}[i], X) \times F(X, S_{n+1}[j]) \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 1 \\
&\quad + \sum_{X \in S_n | X \notin \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 0 \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X) \\
&= \sum_{x \in S_{n+1}[j]} F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) \\
&= \sum_{x \in S_{n+1}[j] | x \in S_{n-1}[i]} F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) + \\
&\quad \sum_{x \in S_{n+1}[j] | x \notin S_{n-1}[i]} F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) \\
&= \sum_{x \in S_{n+1}[j] | x \notin S_{n-1}[i]} F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\})
\end{aligned}
$$

Sketch of the proof

- $S_{n-1}[i] \not\subset S_{n+1}[j]$
- $S_{n-1}[i] \subset S_{n+1}[j]$

Sketch of the proof

- $S_{n-1}[i] \not\subset S_{n+1}[j]$
  $\forall x \in S_{n-1}[i], F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) = 0$
- $S_{n-1}[i] \subset S_{n+1}[j]$

Sketch of the proof

- $S_{n-1}[i] \not\subset S_{n+1}[j]$
  $\forall x \in S_{n-1}[i], F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) = 0$
- $S_{n-1}[i] \subset S_{n+1}[j]$
  $F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) = 1$

$$\begin{aligned} c_{i,j} &= \sum_{x \in S_{n+1}[j] \mid x \notin S_{n-1}[i]} 1 \\ &= \sharp|S_{n+1}[j] \setminus S_{n-1}[i]| \\ &= n + 2 - n = 2 = 0 \mod 2 \end{aligned}$$

Sketch of the proof

- $S_{n-1}[i] \not\subset S_{n+1}[j]$
  $\forall x \in S_{n-1}[i], F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) = 0$
- $S_{n-1}[i] \subset S_{n+1}[j]$
  $F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) = 1$

$$
\begin{aligned}
c_{i,j} &= \sum_{x \in S_{n+1}[j] | x \notin S_{n-1}[i]} 1 \\
&= \sharp|S_{n+1}[j] \setminus S_{n-1}[i]| \\
&= n + 2 - n = 2 = 0 \mod 2
\end{aligned}
$$

$\square$

Table of Contents

SSReflect

- SSReflect:
  - Extension of Coq
  - Developed while formalizing the Four Color Theorem
  - Provides new libraries:

SSReflect

- SSReflect:
  - Extension of Coq
  - Developed while formalizing the Four Color Theorem
  - Provides new libraries:
    - matrix.v: matrix theory
    - finset.v and fintype.v: finite set theory and finite types
    - bigops.v: indexed "big" operations, like $\sum_{i=0}^{n} f(i)$ or $\bigcup_{i \in I} f(i)$
    - zmodp.v: additive group and ring $\mathbb{Z}_p$

Representation of Simplicial Complexes in SSReflect

Definition

Let V be a finite ordered set, called the vertex set, a simplex over V is any finite subset of V.

```
Variable V : finType.
Definition simplex := {set V}.
```

Representation of Simplicial Complexes in SSReflect

Definition

Let V be a finite ordered set, called the vertex set, a simplex over V is any finite subset of V.

Definition

A finite ordered (abstract) simplicial complex over V is a finite set of simplices $\mathcal{K}$ over V satisfying the property:

$$\forall \alpha \in \mathcal{K}, \textit{ if } \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K}$$

```
Variable V : finType.
Definition simplex := {set V}.
Definition simplicial_complex (c : {set simplex}) :=
  forall x, x \in c -> forall y : simplex, y \subset x -> y \in c.
```

Incidence Matrices

Definition

Let X and Y be two ordered finite sets of simplices, we call incidence matrix to a matrix $m \times n$ where

$$m = \sharp|X| \ \wedge \ n = \sharp|Y|$$

$$M = \begin{array}{cc} & \begin{array}{ccc} Y[1] & \cdots & Y[n] \end{array} \\ \begin{array}{c} X[1] \\ \vdots \\ X[m] \end{array} & \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \end{array}$$

$$a_{i,j} = \begin{cases} 1 & \textit{if } X[i] \textit{ is a face of } Y[j] \\ 0 & \textit{if } X[i] \textit{ is not a face of } Y[j] \end{cases}$$

```
Definition face_op (S : simplex) (x : V) := S :\ x.
Definition boundary (S : simplex) := (face_op S) @: S.

Variables Left Top : {set simplex}.
Definition incidenceMatrix :=
  \matrix_(i < #|Left|, j < #|Top|)
      if enum_val i \in boundary (enum_val j) then 1 else 0:'F_2.
```

Incidence Matrices

Definition

Let C be a finite set of simplices, A be the set of n-simplices of C with an order between its elements and B the set of $(n-1)$-simplices of C with an order between its elements.
We call incidence matrix of dimension n ($n \geq 1$), to a matrix $p \times q$ where

$$p = \sharp|B| \ \wedge \ q = \sharp|A|$$

$$M_{i,j} \quad = \quad \begin{cases} 1 & \textit{if } B[i] \textit{ is a face of } A[j] \\ 0 & \textit{if } B[i] \textit{ is not a face of } A[j] \end{cases}$$

```
Section nth_incidence_matrix.
Variable c: {set simplex}.
Variable n:nat.
Definition n_1_simplices := [set x \in c | #|x| == n].
Definition n_simplices := [set x \in c | #|x| == n+1].
Definition incidence_matrix_n :=
  incidenceMatrix n_1_simplices n_simplices.
End nth_incidence_matrix.
```

Product of two consecutive incidence matrices in $\mathbb{Z}_2$

Theorem (Product of two consecutive incidence matrices in $\mathbb{Z}_2$)

Let $\mathcal{K}$ be a finite simplicial complex over V with an order between the simplices of the same dimension and let $n \geq 1$ be a natural number n, then the product of the n-th incidence matrix of $\mathcal{K}$ and the $(n+1)$-incidence matrix of $\mathcal{K}$ over the ring $\mathbb{Z}/2\mathbb{Z}$ is equal to the null matrix.

```
Theorem incidence_matrices_sc_product:
  forall (V:finType) (n:nat) (sc: {set (simplex V)}),
    simplicial_complex sc ->
      (incidence_mx_n sc n) *m (incidence_mx_n sc (n.+1)) = 0.
```

Formalization in SSReflect of the theorem

- Summation part:

Formalization in SSReflect of the theorem

- Summation part:
  - Lemmas from "bigop" library
  - `bigID`: $\sum_{i \in r \mid P_i} F_i = \sum_{i \in r \mid P_i \wedge a_i} F_i + \sum_{i \in r \mid P_i \wedge \sim a_i} F_i$
  - `big1`: $\sum_{i \in r \mid P_i} 0 = 0$

Formalization in SSReflect of the theorem

- Summation part:
  - Lemmas from "bigop" library
  - `bigID`: $\sum_{i \in r \mid P_i} F_i = \sum_{i \in r \mid P_i \wedge a_i} F_i + \sum_{i \in r \mid P_i \wedge \sim a_i} F_i$
  - `big1`: $\sum_{i \in r \mid P_i} 0 = 0$
- Cardinality part:
  - Auxiliary lemmas
  - Lemmas from "finset" and "fintype" libraries

Table of Contents

Conclusions and Further work

- Conclusions:
  - Formalization in Coq/SSReflect:
    - Simplicial complexes
    - Incidence matrices
  - Application of formal methods in software systems

Conclusions and Further work

- Conclusions:
  - Formalization in Coq/SSReflect:
    - Simplicial complexes
    - Incidence matrices
  - Application of formal methods in software systems
- Further work:
  - Formalization:
    - From digital images to simplicial complexes
    - Computation Smith Normal Form
  - $\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}$
  - Executability of the proofs:
    - Code extraction
    - Internal computations

Incidence Simplicial Matrices Formalized in Coq/SSReflect*

Jónathan Heras, María Poza, Maxime Dénès, and Laurence Rideau

Universidad de La Rioja, Spain - INRIA Sophia Antipolis (Méditerranée)

CICM 2011, Calculemus track,
July 22, 2011

* Partially supported by Ministerio de Educación y Ciencia, project MTM2009-13842-C02-01, and by European Commission FP7, STREP project ForMath, n. 243847